Apollonian Circle Packings: Number Theory
II. Spherical and Hyperbolic Packings

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ABSTRACT

Apollonian circle packings arise by repeatedly filling the interstices between mutually tangent circles with further tangent circles. In Euclidean space it is possible for every circle in such a packing to have integer radius of curvature, and we call such a packing an *integral Apollonian circle packing.* There are infinitely many different integral packings; these were studied in the paper [8]. Integral circle packings also exist in spherical and hyperbolic space, provided a suitable definition of curvature is used (see [10]) and again there are an infinite number of different integral packings. This paper studies number-theoretic properties of such packings. This amounts to studying the orbits of a particular subgroup $\mathcal{A}$ of the group of integral automorphs of the indefinite quaternary quadratic form $Q_{\mathcal{D}}(w,x,y,z) = 2(w^2+x^2+y^2+z^2)-(w+x+y+z)^2$. This subgroup, called the Apollonian group, acts on integer solutions $Q_{\mathcal{D}}(w,x,y,z) = k$. This paper gives a reduction theory for orbits of $\mathcal{A}$ acting on integer solutions to $Q_{\mathcal{D}}(w,x,y,z) = k$ valid for all integer k. It also classifies orbits for all $k \equiv 0 \pmod 4$ in terms of an extra parameter n and an auxiliary class group (depending on n and k), and studies congruence conditions on integers in a given orbit.

Keywords: Circle packings, Apollonian circles, Diophantine equations, Lorentz group
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1. Introduction

A Descartes configuration is a set of four mutually touching circles with distinct tangents. A (Euclidean) Apollonian circle packing in the plane is constructed starting from a Descartes configuration by recursively adding circles tangent to three of the circles already constructed in the packing; see [10], [5], [6], and Figure 1.

Apollonian packings can be completely described by the set of Descartes configurations they contain. The curvatures $(a,\ b,\ c,\ d)$ of the circles in a Descartes configuration satisfy an algebraic relation called the Descartes circle theorem. Using the *Descartes quadratic form*

$$Q_{\mathcal{D}}(a,b,c,d) := 2(a^2+b^2+c^2+d^2) - (a+b+c+d)^2, \tag{1.1}$$

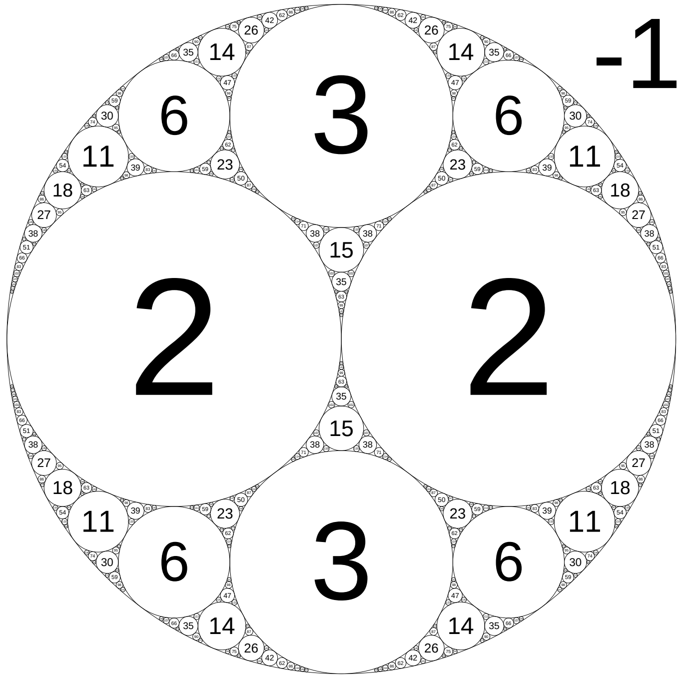

Figure 1: The Euclidean Apollonian circle packing (−1,2,2,3)

the Descartes circle theorem states that

$$Q_{\mathcal{D}}(a,b,c,d) = 0. \tag{1.2}$$

Descartes originally stated this relation in another algebraic form; see [10]. In order to have this formula work in all cases, the curvatures of the circles in a Descartes configuration must be given proper signs, as specified in [10] and [5].

The form $Q_{\mathcal{D}}$ is an integral quadratic form in four variables, and the equation (1.2) has many integer solutions. If the initial Descartes configuration generating a (Euclidean) Apollonian packing has all four circles with integral curvatures, then it turns out that all the circles in the packing have integer curvatures. This integer structure was studied in the first paper in this series [8] using the action of a particular discrete subgroup of automorphisms of the Descartes quadratic form, which was there termed the *Apollonian group* $\mathcal{A}$. It is a group of integer matrices, with generators explicitly given in §3. The Descartes configurations in a (Euclidean) Apollonian packing are described by an orbit of the Apollonian group. That is, the Apollonian group acts on the column vector of curvatures of the initial Descartes configuration, and generates the curvature vectors of all Descartes configurations in the packing. This fact explains the preservation of integrality of the curvatures. The Apollonian group and its relevance for Apollonian packings was already noted in 1992 by Söderberg [19].

The first paper in this series [8] studied Diophantine properties of the curvatures in Euclidean integer Apollonian circle packings, based on the Apollonian group action. In [10] it was observed that there exist notions of integer Apollonian packings in spherical and hyperbolic geometry. These are based on analogues of the Descartes equation valid in these geometries, as follows. In hyperbolic space, if the "curvature" of a circle with hyperbolic radius r is taken to be $\coth(r)$, then the modified Descartes equation for the "curvatures" of circles in a Descartes configuration is

$$Q_{\mathcal{D}}(a, b, c, d) = 4. \tag{1.3}$$

Similarly, in spherical space, if the "curvature" of a circle with spherical radius r is defined to be $\cot(r)$, then the modified Descartes equation is

$$Q_{\mathcal{D}}(a, b, c, d) = -4. \tag{1.4}$$

There exist integral Apollonian packings in both these geometries, in the sense that the (spherical or hyperbolic) "curvatures" of all circles in the packing are integral; see the examples in Figures 2 and 3 below. Again, the vectors of "curvatures" of all Descartes configurations in such a packing form an orbit under the action of the Apollonian group.

The purpose of this paper is to study Diophantine properties of the orbits of the Apollonian group $\mathcal{A}$ acting on the integer solutions to the Diophantine equation

$$Q_{\mathcal{D}}(a, b, c, d) = k, \tag{1.5}$$

where k is a fixed integer. Integer solutions exist for $k \equiv 0$ or $3 \pmod 4$, and our results on reduction theory in §3 are proved in this generality. Our other results are established under the more restrictive condition $k \equiv 0 \pmod 4$, where we will always write $k = 4m$. These results cover the motivating cases $k = \pm 4$ corresponding to spherical and hyperbolic Apollonian packings, as well as the Euclidean case $k = 0$. We note that the Apollonian group $\mathcal{A}$ is a subgroup of *infinite* index in the group $\mathrm{Aut}(Q_{\mathcal{D}}, \mathbb{Z})$ of all integer automorphs of the Descartes quadratic form $Q_{\mathcal{D}}$. As a consequence, the integral solutions of $Q_{\mathcal{D}}(a, b, c, d) = k$ (for $k \equiv 0$ or $3 \pmod 4$) fall in an infinite number of orbits under the action of $\mathcal{A}$. In contrast, these same integral points fall in a finite number of orbits under the full $\mathrm{Aut}(Q_{\mathcal{D}}, \mathbb{Z})$-action.

In §2, we study the distribution of integer solutions to the equation $Q_{\mathcal{D}}(a, b, c, d) = 4m$, determining asymptotics for the number of such solutions to these equations of size below a given bound. These asymptotics are obtained via a bijective correspondence between integral representations of integers $4m$ by the Descartes form and representations of integers $2m$ by the Lorentzian form $Q_{\mathcal{L}}(W, X, Y, Z) = -W^2 + X^2 + Y^2 + Z^2$, and then applying results of Ratcliffe and Tschantz [17] on integer solutions to the Lorentzian form.

In §3, we describe the Apollonian group, and apply it to give a reduction algorithm for quadruples $\mathbf{v} = (a, b, c, d)$ satisfying $Q_{\mathcal{D}}(\mathbf{v}) = k$ in a fixed orbit of the Apollonian group, starting from an initial integer quadruple satisfying $Q_{\mathcal{D}}(\mathbf{v}) = k$. This reduction algorithm differs in appearance from that given in part I [8, Sect. 3], which treated the case $k = 0$; however we show its steps coincide with those of the earlier algorithm when $k = 0$. A quadruple obtained at the end of this algorithm is called a *reduced quadruple.* We classify reduced quadruples as two types, *root quadruples* and *exceptional quadruples.* We show that for each orbit containing a root quadruple, the root quadruple is the unique reduced quadruple in that orbit. For each fixed k we show that there are at most a finite number of orbits not containing a root quadruple. Each of these orbits contain at least one and at most finitely many reduced quadruples, each of which is by definition an exceptional quadruple. We show that such orbits can only occur when $k > 0$, and we call them *exceptional orbits.* Their existence is an interesting new phenomenon

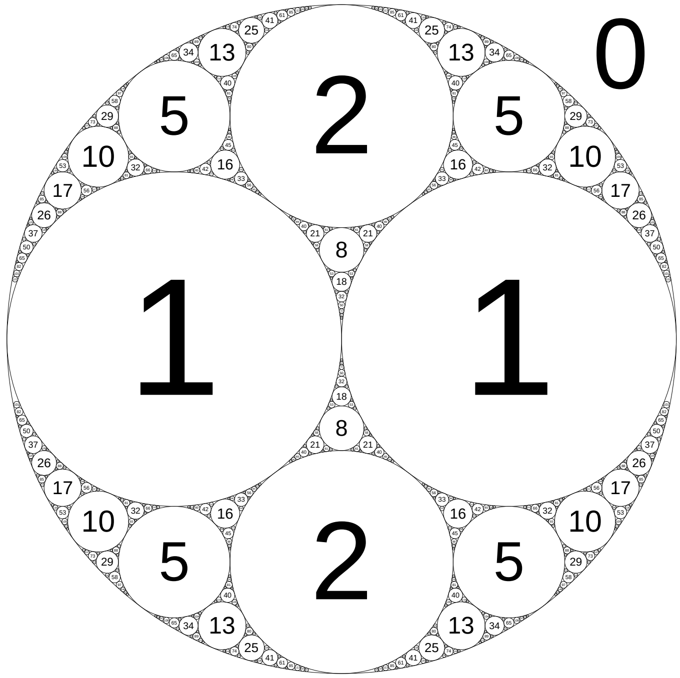

Figure 2: The spherical Apollonian circle packing (0,1,1,2)

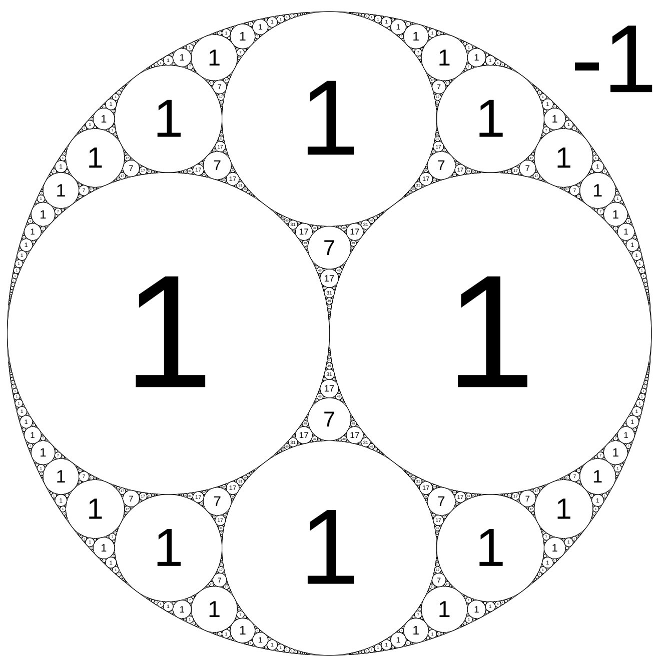

Figure 3: The hyperbolic Apollonian circle packing $(-1,1,1,1)$

uncovered in this work. They do occur in the hyperbolic Apollonian packing case $k = 4$, where we show there are exactly two exceptional orbits, labeled by the hyperbolic Descartes quadruples $(0, 0, 0, 2)$ and $(-1, 0, 0, 1)$, respectively. We give a geometric interpretation of these exceptional hyperbolic Apollonian packings.

As examples, Figure 2 shows the spherical ($k = -4$) Apollonian packing with root quadruple $(0, 1, 1, 2)$, after stereographic projection. Figure 3 shows the hyperbolic ($k = 4$) Apollonian packing with root quadruple $(-1, 1, 1, 1)$, using the unit disc model of the hyperbolic plane. For more on the geometry of Apollonian packings in spherical and hyperbolic space consult [10].

In §4, we count the number $N_{\text{root}}(4m, -n)$ of root quadruples $\mathbf{a} = (a, b, c, d)$ with $Q_{\mathcal{D}}(\mathbf{a}) = 4m$ having smallest member $a = -n$. We give a formula for this number in terms of the class number of (not necessarily primitive) integral binary quadratic forms of discriminant $-4(n^2 - m)$ under $GL(2, \mathbb{Z})$-equivalence. The class number interpretation applies only when $n^2 > m$. For certain $m > 0$ there may exist root quadruples with parameters $n^2 \leq m$, and information on them is given in Theorem 3.3 in §3. For the hyperbolic case the only value not covered is $n = -1$. Associated to it is an infinite family of distinct root quadruples $\{(-1, 1, c, c) |\ c \geq 1\}$. Figure 3 pictures the Apollonian packing with root quadruple $(-1, 1, 1, 1)$.

The class number equivalence allows us to derive good upper bounds for the number of such root quadruples with smallest value $a = -n$ as $n \to \infty$, namely

$$N_{\text{root}}(4m, -n) = O\left(n(\log n)(\log\log n)^2\right).$$

If $m = \pm 1$, then these bounds are interpretable as bounds on the integral spherical and hyperbolic packings with boundary an enclosing circle of (spherical or hyperbolic) curvature n.

In §5 we investigate congruence restrictions on the integer curvatures which occur in a packing, for the case of spherical and hyperbolic packings, i.e., $k = \pm 4$. We show that there are always non-trivial congruence conditions modulo 12 on the set of integers that can occur in a given packing. It seems reasonable to expect that such congruence conditions can only involve powers of the primes 2 and 3 and that all sufficiently large integers which are not excluded by these congruence conditions actually occur.

Various open problems are raised at the end of §4 and in §5. In addition, in the case of Euclidean Apollonian packings, it is known that there are Apollonian packings which have stronger integer properties, involving the centers of the circles as well as the curvatures. These were studied in Graham et al. [5], [6], and [7]. It remains to be seen if there are analogues of such properties (for circle centers) in the spherical and hyperbolic cases.

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2. Integral Descartes Quadruples

The *Descartes quadratic form* $Q_{\mathcal{D}}$ is the quaternary quadratic form

$$Q_{\mathcal{D}}(w, x, y, z) = 2(w^2 + x^2 + y^2 + z^2) - (w + x + y + z)^2. \tag{2.1}$$

Its matrix representation, for $\mathbf{v} = [w, x, y, z]^T$, is

$$Q_{\mathcal{D}}(\mathbf{v}) = \mathbf{v}^T \mathbf{Q}_{\mathcal{D}} \mathbf{v} = \mathbf{v}^T \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \mathbf{v}.$$

This form is indefinite with signature $(+, +, +, -)$. We consider integral representations of an integer k by this quaternary quadratic form. For an integer quadruple $\mathbf{v} = (w, x, y, z) \in \mathbb{Z}^4$, $Q_{\mathcal{D}}(w, x, y, z) \equiv 0$ or $3 \pmod 4$, according to the parity of $w + x + y + z$. We define the *Euclidean height* $H(\mathbf{v})$ of a (real) quadruple $\mathbf{v} = (w, x, y, z)$ to be

$$H(\mathbf{v}) := (w^2 + x^2 + y^2 + z^2)^{1/2}.$$

Notice that we reserve the usual notation, $|v|$, for another meaning in §3.

We relate integer representations of even k of the Descartes form to integer representations of $\frac{k}{2}$ of the *Lorentz form*

$$Q_{\mathcal{L}}(W, X, Y, Z) = -W^2 + X^2 + Y^2 + Z^2.$$

in Lemma 2.1 below. The relation only works for even k, so in this section we consider only the case $k \equiv 0 \pmod 4$, so that

$$Q_{\mathcal{D}}(w, x, y, z) = k = 4m. \tag{2.2}$$

The Lorentz form has the matrix representation, for $\mathbf{v} = [W, X, Y, Z]^T$,

$$Q_{\mathcal{L}}(\mathbf{v}) = \mathbf{v}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{v}.$$

The Descartes form and Lorentz form are related by

$$2Q_{\mathcal{L}} = \mathbf{J}_0 Q_{\mathcal{D}} \mathbf{J}_0^T, \tag{2.3}$$

where

$$\mathbf{J}_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Notice that $\mathbf{J}_0^2 = \mathbf{I}$.

Let $N_{\mathcal{L}}(k, T)$ count the number of integer representations of k by the Lorentz form of Euclidean height at most T and let $N_{\mathcal{D}}(k, T)$ denote the number of solutions of (2.2) of Euclidean height at most T.

Lemma 2.1. *The mapping $(W, X, Y, Z)^T = \mathbf{J}_0(w, x, y, z)^T$ gives a bijection between real solutions of $Q_{\mathcal{D}}(w, x, y, z) = 4r$ and real solutions of $Q_{\mathcal{L}}(W, X, Y, Z) = 2r$ for each $r \in \mathbb{R}$, and this bijection preserves the Euclidean height of solutions. In the case where $r = m \in \mathbb{Z}$ this bijection restricts to a bijection between integer representations of $4m$ by the Descartes form and integer representations of $2m$ by the Lorentzian form. In particular, for all integers m,*

$$N_{\mathcal{D}}(4m, T) = N_{\mathcal{L}}(2m, T).$$

Proof. There is a bijection on the level of real numbers, by (2.3), since $\mathbf{J}_0$ is invertible, and $Q_{\mathcal{D}}(w,x,y,z) = 4r$ if and only if $Q_{\mathcal{L}}(W,X,Y,Z) = 2r$. A calculation using $\mathbf{J}_0^2 = \mathbf{I}$ shows that

$$H(w,x,y,z) = w^2 + x^2 + y^2 + x^2 = W^2 + X^2 + Y^2 + Z^2 = H(W,X,Y,Z).$$

Now suppose that $r = m \in \mathbb{Z}$. The mapping takes integral solutions (w,x,y,z) to integral solutions (W,X,Y,Z) because all integral solutions of $Q_{\mathcal{D}}(w,x,y,z) = 4m$ satisfy $w + x + y + z \equiv 0 \pmod 2$, as can be seen by reducing the Descartes equation (2.2) modulo 2. In the reverse direction integral solutions go to integral solutions because all solutions of $Q_{\mathcal{L}}(w,x,y,z) = 2m$ have $W + X + Y + Z \equiv 0 \pmod 2$, similarly. □

Representation of integers by the Lorentz form has been much studied; see Radcliffe and Tschantz [17]. Their results immediately yield the following result.

Theorem 2.2. *For each nonzero integer m, there is an explicitly computable rational number $c(2m)$ such that*

$$N_{\mathcal{D}}(4m,T) = c(2m)\frac{\pi}{L(2,\chi_{-4})}T^2 + o(T^2), \tag{2.4}$$

in which

$$L(2,\chi_{-4}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159.$$

We have $c(-2) = \frac{3}{4}$ and $c(2) = \frac{5}{4}$.

Proof. From Lemma 2.1 the problem of counting integral solutions to $Q_{\mathcal{D}}(w,x,y,z) = 4m$ of height at most T is equivalent to counting the number of integral solutions to $Q_{\mathcal{L}}(W,X,Y,Z) = 2m$ of Euclidean height at most T. We use asymptotic formulae of Radcliffe and Tschantz for the number $r(n,k,T)$ of solutions to the Lorentz form $-X_0^2 + X_1^2 + \cdots + X_n^2 = k$ with $|X_0| \leq T$, for any $n \geq 2$ for any fixed nonzero k. We take $n = 3$, and their formula for $k < 0$ in [17, Theorem 3, p. 505] is

$$r(3,k,T) = \frac{\mathrm{Vol}(S^2)}{2}\delta(3,k)T^2 + O(T^{3/2}), \tag{2.5}$$

in which $\mathrm{Vol}(S^2) = \pi$ and $\delta(3,k)$ is a density constant for the representation in the sense of Siegel [18]. We need asymptotics for the number of solutions $s(n,k,T)$ to the Lorentz form $Q_{\mathcal{L}}(W,X,Y,Z) = k$ of Euclidean height below T, which is given by

$$s(n,k,T) := r\left(n, k, \sqrt{\frac{1}{2}(T^2 - k)}\right).$$

The asymptotic formula (2.5) yields

$$s(3,k,T) = \frac{\mathrm{Vol}(S^2)}{4}\delta(3,2m)T^2 + O(T^{3/2}). \tag{2.6}$$

Their result [17, Theorem 12, p. 518] evaluates the constant $\delta(3,k)$ in terms of the value of an L-function, as

$$\delta(3,k) = \prod_{p \mid k} \left(\frac{\delta_p(3,k)}{(1-(\frac{-4}{p})p^{-2})} \right) \cdot \frac{1}{L(2,\chi_{-4})},$$

in which each $\delta_p(3,k)$ is an effectively computable local density which is a rational number. We now set $k = 2m$, and obtain a formula of the desired shape (2.4), with

$$c(2m) = \frac{1}{4} \left(\frac{\delta_p(3,2m)}{(1-(\frac{-4}{p})p^{-2})} \right).$$

Radcliffe and Tschantz [17, Theorem 4, p. 510] also obtained an asymptotic formula for solutions to the Lorentz form with $k > 0$, which gives

$$r(3,k,T) = \frac{\mathrm{Vol}(S^2)}{2} \delta(3,k) T^2 + o(T^2).$$

We derive a similar asymptotic formula for $s(n,k,T)$ (without explicit error term), by setting $k = 2m$, and proceeding as above.

The particular values $c(-2) = \frac{3}{4}$ and $c(2) = \frac{5}{4}$ are derived from Table II in Radcliffe and Tschantz [17, p. 521], using $k = \pm 2$. $\square$

Remarks. (1) The case $k = 0$ exhibits a significant difference from the pattern of Theorem 2.2. In [8, Theorem 2.1] it was shown that

$$N_{\mathcal{D}}(0,T) = \frac{1}{4} \frac{\pi^2}{L(2,\chi_{-4})} T^2 + O(T(\log T)^2).$$

That is, the coefficient $c(0) = \frac{\pi}{4}$ is *transcendental.*

(2) Theorem 2.2 implies that the number of hyperbolic Descartes quadruples of Euclidean height below T is asymptotically $\frac{5}{3}$ the number of spherical Descartes quadruples of height below T, as $T \to \infty$. That is,

$$\lim_{T \to \infty} \frac{N_{\mathcal{D}}(4,T)}{N_{\mathcal{D}}(-4,T)} = \frac{5}{3}. \tag{2.7}$$

3. Reduction Theory and Root Quadruples

The results of this section apply to the general case

$$Q_{\mathcal{D}}(w,x,y,z) = k, \tag{3.1}$$

with $k \equiv 0$ or $3 \pmod 4$ an integer. We introduce the Apollonian group, and present a reduction procedure which, given an integral Descartes quadruple, uses the action of the Apollonian group to transform it to a reduced quadruple, which is minimal according to a certain measure. The reduced quadruples then classify integral Apollonian circle packings, generally uniquely, but up to a finite ambiguity in some exceptional cases.

3.1. Apollonian Group

In [8] and [10] Apollonian circle packings were specified in terms of the set of Descartes configurations they contain. This set of configurations completely describes the packing, and it was observed there that, in a suitable coordinate system, they form as the orbit of any single Descartes configuration in the packing under the motion of a discrete group, the Apollonian group.

Definition 3.1. The *Apollonian group* $\mathcal{A} = \langle \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4 \rangle$ is the subgroup of $GL(4, \mathbb{Z})$ generated by the four integer 4×4 matrices

$$\mathbf{S}_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{S}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\mathbf{S}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{S}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

The Apollonian group is a subgroup of the integer automorphism group $\mathrm{Aut}(Q_{\mathcal{D}}, \mathbb{Z})$ of the Descartes form under congruence, which is the largest subgroup of $GL(4, \mathbb{Z})$ which leaves $Q_{\mathcal{D}}$ invariant; that is,

$$\mathbf{U}^T \mathbf{Q}_{\mathcal{D}} \mathbf{U} = \mathbf{Q}_{\mathcal{D}}, \qquad \text{for all } \mathbf{U} \in \mathcal{A}, \tag{3.2}$$

a relation which needs only be checked on the four generators $\mathbf{S}_i$. In particular it preserves the level sets $Q_{\mathcal{D}}(\mathbf{v}) = k$. It acts on all (real) Descartes quadruples $\mathbf{v} = (a, b, c, d)^T$ (viewed as column vectors) via matrix multiplication, sending $\mathbf{v}$ to $U\mathbf{v}$.

The group $\mathrm{Aut}(Q_{\mathcal{D}}, \mathbb{Z})$ is very large, and can be put in one-to-one correspondence with a finite index subgroup of the integer Lorentzian group $O(3, 1, \mathbb{Z})$ (using the correspondence (2.3) in §2). The Apollonian group $\mathcal{A}$ is of infinite index in $\mathrm{Aut}(Q_{\mathcal{D}}, \mathbb{Z})$, and it has infinitely many distinct integral orbits for each integer $k \equiv 0$ or $3 \pmod 4$, the values for which integral solutions exist.

Geometrically, the generators $\mathbf{S}_i$ in $\mathcal{A}$ correspond to inversions with respect to a circle passing through three intersection points in a Descartes configuration. This inversion gives a new Descartes configuration in the same packing, leaving three of the circles fixed; see [5, Section 2]. In particular the group can be viewed as acting on column vectors $\mathbf{v} = [a, b, c, d]^T$ giving the "curvatures" in a Descartes configuration, converting it to the curvatures of another Descartes configuration in the packing. The generators $\mathbf{S}_i$ allow us to move around inside the packing, from one Descartes configuration to a neighboring configuration. From (2.1) and (3.1) we see that if the curvatures a, b, c of three touching circles are given, then the curvatures of the two possible circles which are tangent to these three satisfy

$$d, d' = a + b + c \pm \sqrt{4(ab + bc + ac) + k}.$$

Therefore $d + d' = 2(a + b + c)$. If we begin with circles with curvatures a, b, c, d, the curvature $d'$of the other circle which touches the first three is

$$d' = 2(a + b + c) - d.$$

The Apollonian group action corresponding to this sends (a, b, c, d) to (a, b, c, d'), which is the action of $\mathbf{S}_4$. In particular, if we start with an integer quadruple, this procedure will only create other integer quadruples. This group is studied in Aharonov [1], Graham et al. [5], [6] and Söderberg [19].

Definition 3.2. *An* integral Descartes ensemble $\mathcal{A}[\mathbf{v}]$ *is an orbit of the Apollonian group* $\mathcal{A}$ *acting on an integral quadruple* $\mathbf{v} = (w, x, y, z)^T \in \mathbb{Z}^4$. *That is,* $\mathcal{A}[\mathbf{v}] := \{\mathbf{U}\mathbf{v} : \ \mathbf{U} \in \mathcal{A}\}$.

All elements $\mathbf{x} \in \mathcal{A}[\mathbf{v}_0]$ are integer quadruples satisfying

$$Q_{\mathcal{D}}(\mathbf{x}) = Q_{\mathcal{D}}(\mathbf{v}_0) = k.$$

We are most interested in the case $L(\mathbf{v}) := w + x + y + z \equiv 0 \pmod 2$, where $k = 4m$. For the cases $m = -1, 1$ such ensembles give the set of Descartes quadruples in an integral spherical (resp. hyperbolic) Apollonian circle packing.

This section addresses the problem of classifying the orbits of the Apollonian group acting on the set of all integral solutions $Q_{\mathcal{D}}(\mathbf{v}) = k$ for a fixed value of k.

3.2. General Reduction Algorithm

We present a reduction procedure which, given an integral quadruple $\mathbf{v}_0$, finds an element of (locally) "minimal" size in the orbit $\mathcal{A}[\mathbf{v}_0]$. We will show that every orbit contains at least one and at most finitely many reduced elements. In most cases the "minimal" quadruple is unique and is independent of the starting point $\mathbf{v}_0$ of the reduction procedure, and this is the case for root quadruples defined below. However when $k > 0$ there sometimes exist exceptional integral Descartes ensembles containing more than one reduced element, and the outcome of the reduction procedure depends on its starting point in the orbit. We show that for each fixed k there are in total at most finitely many such exceptional orbits.

The general reduction procedure given below greedily attempts to reduce the size of the elements in a Descartes quadruple $\mathbf{v} := (a, b, c, d)^T$ by applying the generators $\mathbf{S}_i$ to decrease the quantity

$$|\mathbf{v}| := |a| + |b| + |c| + |d|.$$

General reduction algorithm.
Input: An integer quadruple $(a,\ b,\ c,\ d)$.

(1) Order the quadruple so that $a \le b \le c \le d$. *Then test in order* $1 \le i \le 4$ *whether some* $\mathbf{S}_i$ *decreases* $|\mathbf{v}| := |a| + |b| + |c| + |d|$. *For the first* $\mathbf{S}_i$ *that does, apply it to produce a new quadruple, and continue.*

(2) If no $\mathbf{S}_i$ *strictly decreases* $|\mathbf{v}|$, *halt. Declare the result a* reduced quadruple.

This reduction procedure is slightly different from the reduction algorithm used in part I [8] for the case $k = 0$. The part I reduction procedure applied only to quadruples with $L(\mathbf{a}) = a + b + c + d > 0$, and tried to decrease the invariant $L(\mathbf{a}) = a + b + c + d$ at each step, halting if this could not be done. Although the general reduction algorithm uses a different reduction rule, one can show in this case that it takes the identical sequence of steps as the reduction algorithm in part I; see the remark at the end of §3.3. Thus the general reduction algorithm can be viewed as a strict generalization of the algorithm of [8].

We will classify reduced quadruples as either *root quadruples* or *exceptional quadruples*, as defined in the following theorem.

Theorem 3.1. *The general reduction algorithm starting from any nonzero integer quadruple* $\mathbf{a} = (a, b, c, d)$ *always halts in a finite number of steps. Let the reduced quadruple at termination be ordered as* $\mathbf{a} = (a, b, c, d)$ *with* $a \leq b \leq c \leq d$*. Suppose* $L(\mathbf{a}) = a + b + c + d \geq 0$*. Then exactly one of the following holds:*

(i) The quadruple $\mathbf{a}$ *satisfies*

$$a + b + c \geq d > 0. \tag{3.3}$$

In this case we call $\mathbf{a}$ *a root quadruple.*

(ii) The quadruple $\mathbf{a}$ *satisfies*

$$a + b + c \leq 0 < d. \tag{3.4}$$

In this case we call $\mathbf{a}$ *an exceptional quadruple.*

If $\mathbf{a}$ *is a reduced quadruple such that* $L(\mathbf{a}) < 0$*, then* $\mathbf{a}^* := (-d, -c, -b, -a)$ *is a reduced quadruple with* $L(\mathbf{a}^*) > 0$*. We call such* $\mathbf{a}$ *a root quadruple (resp. exceptional quadruple) if and only if* $\mathbf{a}^*$ *is a root quadruple (resp. exceptional quadruple).*

Proof. The integer-valued invariant $|\mathbf{a}| := |a| + |b| + |c| + |d|$ strictly decreases at each step of the reduction algorithm. It is nonnegative, so the process stops after finitely many iterations.

Now suppose that $\mathbf{a}$ is reduced and $L(\mathbf{a}) = a + b + c + d \geq 0$. We must show that (i) or (ii) holds. Now $d > 0$ because $\mathbf{a}$ is nonzero. The condition that $\mathbf{a}$ be reduced requires $|\mathbf{S}_4\mathbf{a}| \geq |\mathbf{a}|$, and since $\mathbf{S}_4\mathbf{a} = (a, b, c, d')$ with $d' = 2(a + b + c) - d$, this condition is $|d'| \geq |d| = d$. Now $d' \geq d$ gives $a + b + c \geq d$ which is (i), while $d' \leq -d$ gives $a + b + c \leq 0$ which is (ii). □

The definition of root quadruple formulated in Theorem 3.1 is not identical in form to the definition of root quadruple used in part I [8] for the case $k = 4m = 0$. However they are equivalent definitions, since the reduction algorithms take identical steps, as remarked above.

3.3. Root Quadruples

We first show that the reduction algorithm is well behaved for all orbits $\mathcal{A}[\mathbf{v}]$ containing a root quadruple.

Theorem 3.2. *Let* $\mathcal{A}[\mathbf{v}]$ *be a nonzero integer orbit of the Apollonian group, and suppose that* $\mathcal{A}[\mathbf{v}]$ *contains a root quadruple* $\mathbf{a}$*. Then:*

(1) The quadruple $\mathbf{a}$ *is the unique reduced quadruple in* $\mathcal{A}[\mathbf{v}]$.

(2) All values $L(\mathbf{x})$ *for* $x \in \mathcal{A}[\mathbf{v}]$ *are nonzero and have the same sign. If this sign is positive, then the general reduction algorithm starting from any* $\mathbf{x} \in \mathcal{A}[\mathbf{v}]$ *applies only* $\mathbf{S}_4$ *until the root quadruple is reached.*

Proof. It suffices to prove the result in the case when the root quadruple has $L(\mathbf{a}) = a+b+c+d \geq 0$, because the case $L(\mathbf{a}) < 0$ can be treated using $\mathbf{a}^*$. Without loss of generality we may reorder the quadruple $a \leq b \leq c \leq d$ (since $\mathcal{A}$ is invariant under conjugation by elements of the symmetric group on 4 letters, represented as permutation matrices). Now the root quadruple condition (3.3) yields

$$L(\mathbf{a}) = a + b + c + d > a + b + c \geq d > 0. \tag{3.5}$$

This implies that

$$a + b \geq d - c \geq 0, \tag{3.6}$$

where the last inequality follows from the ordering.

Claim. *Let* $\mathbf{b} \in \mathcal{A}[\mathbf{v}]$ *be a Descartes quadruple and let* $\mathbf{P}_\sigma \mathbf{b} = (w, x, y, z)$ *be its coordinates permuted into increasing order* $w \leq x \leq y \leq z$. *Then these coordinates satisfy:*

$$\textit{Property (P1):} \qquad w + x \geq 0.$$

If $\mathbf{b}$ *does not equal the given root quadruple* $\mathbf{a}$ *then these coordinates satisfy:*

$$\textit{Property (P2):} \quad 0 < w + x + y < z.$$

To prove the claim, we observe that every quadruple $\mathbf{b} \neq \mathbf{a}$ in $\mathcal{A}[\mathbf{v}]$ can be written as $\mathbf{b} = \mathbf{S}_{i_k}\mathbf{S}_{i_{k-1}} \cdots \mathbf{S}_{i_1}\mathbf{a}$ with each $\mathbf{S}_{i_j} \in \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4\}$, for some $k \geq 1$, and with $\mathbf{S}_{i_j} \neq \mathbf{S}_{i_{j-1}}$ for $2 \leq j \leq k$, since all $\mathbf{S}_i^2 = I$. We proceed by induction on the number of multiplications k.

The case $k = 0$ is the root quadruple $\mathbf{a}$, which satisfies property (P1) by (3.6). For the base case $k = 1$ we must verify that properties (P1) and (P2) hold for $\mathbf{S}_j\mathbf{a}$ for $1 \leq j \leq 4$, where $\mathbf{a}$ is the given root quadruple. Without loss of generality we may assume $\mathbf{a} = (a, b, c, d)$ has the ordering $a \leq b \leq c \leq d$; a permutation of $\mathbf{a}$ merely permutes the four values $\mathbf{S}_j\mathbf{a}$. Consider first $\mathbf{S}_1\mathbf{a} = (a', b, c, d)$. We assert that the ordering of the elements of $\mathbf{S}_1\mathbf{a}$ is

$$b \leq c \leq d < a'. \tag{3.7}$$

To establish this we first show

$$a' := 2(b + c + d) - a = (b + c + d) + (b + c + d - a) > b + c + d. \tag{3.8}$$

This inequality holds since $b - a \geq 0$ from the ordering while $d > 0$ and $c \geq 0$ from (3.5). This yields $a' > b + c + d \geq (a + b) + d \geq d$, which establishes (3.7). Property (P1) for $\mathbf{S}_1\mathbf{a}$ asserts that $b + c \geq 0$, and this holds by (3.6). Property (P2) for $\mathbf{S}_1\mathbf{a}$ asserts that $0 < b + c + d < a'$, and the left inequality follows from $b + c + d \geq (a + b) + d > 0$, using both (3.5) and (3.6), while the right inequality follows from (3.8). This proves the base case for $\mathbf{S}_1\mathbf{a}$. The proofs of the base cases $\mathbf{S}_j\mathbf{a}$ for $j = 2, 3$ are essentially the same as for $\mathbf{S}_1\mathbf{a}$. In these cases the one curvature that changes becomes strictly larger than d. Finally we treat $\mathbf{S}_4\mathbf{a} = (a, b, c, d')$. The ordering that holds here is

$$a \le b \le c \le d'.$$

Indeed we have

$$d' = 2(a+b+c) - d = (a+b+c) + (a+b+c-d) \ge a+b+c \ge d, \tag{3.9}$$

by (3.5). If $d = d'$ we stay at the root quadruple, so we may assume $d' > d$ in what follows. Property (P1) for $\mathbf{S}_4\mathbf{a}$ asserts $a + b \ge 0$, and this holds by (3.6). Property (P2) for $\mathbf{S}_4\mathbf{a}$ asserts $0 < a + b + c < d'$. Now if $a + b + c = d$ then $d' = d$, so assuming $d' > d$ we must have $a+b+c > d$. Using this fact, we see that the inequalities in (3.9) are both strict, which gives the right inequality $a + b + c < d'$. The left inequality $a + b + c > 0$ holds by (3.5). Thus properties (P1) and (P2) hold for $\mathbf{S}_4\mathbf{a}$ as well, or else we stay at the root quadruple. This verifies (P1) and (P2) in all cases, and shows that if $\mathbf{b} = \mathbf{S}_j\mathbf{a} \ne \mathbf{a}$, then the largest coordinate in $\mathbf{b}$ is unique and is the coordinate in the j-th position.

For the induction step, for $k+1$ given k, we suppose that $\mathbf{b} = \mathbf{S}_{i_k}\mathbf{S}_{i_{k-1}} \cdots \mathbf{S}_{i_1}\mathbf{a}$ with $\mathbf{b} \ne \mathbf{a}$, $k \ge 1$, and we include in the induction hypothesis the assertion that the largest coordinate of $\mathbf{b}$ is unique and occurs in the i_k-th coordinate position. Let $\mathbf{b}' = (w, x, y, z) = \mathbf{P}_\sigma\mathbf{b}$ denote its rearrangement in increasing order $w \le x \le y \le z$, where $\mathbf{P}_\sigma$ is a 4×4 permutation matrix. The induction hypothesis asserts that (P1) and (P2) hold for $\mathbf{b}'$, together with the strict inequality $y < z$. The induction step requires showing that properties (P1) and (P2) hold for $\mathbf{S}_j\mathbf{b}'$ for $1 \le j \le 3$, corresponding to a product of length $k + 1$, and that the new coordinate formed in $\mathbf{S}_j\mathbf{b}'$ is its strictly largest coordinate. The case of multiplication by $\mathbf{S}_4$ is excluded because it corresponds to choosing $\mathbf{S}_{i_{k+1}} = \mathbf{S}_{i_k}$. Indeed $\mathbf{S}_4$ changes the largest coordinate, which $\mathbf{S}_{i_k}$ did at the previous step by the induction hypothesis, so that $\mathbf{S}_{i_{k+1}} := \mathbf{P}_\sigma^{-1}\mathbf{S}_4\mathbf{P}_\sigma = \mathbf{S}_{i_k}$. The first case is $\mathbf{S}_1\mathbf{b}' = (w', x, y, z)$. We assert that

$$x \le y \le z < w'. \tag{3.10}$$

Indeed $x, y, z \ge 0$ by property (P1) for $\mathbf{b}'$ and

$$w' = 2(x+y+z) - w = (x+y+z) + (x+y+z-w) > x+y+z,$$

since property (P2) for $\mathbf{b}'$ gives $x + y + z > w + 2x + 2y \ge w$. Next property (P1) for $\mathbf{b}'$ gives

$$x + y + z \ge (w + x) + z \ge z > y \ge 0,$$

whence $w' > x + y + z \ge z$ and (3.10) follows. Property (P1) for $\mathbf{S}_1\mathbf{b}'$ asserts $x + y \ge 0$, which holds since $x + y \ge w + x \ge 0$, using Property (P1) for $\mathbf{b}'$. Property (P2) for $\mathbf{S}_1\mathbf{b}'$ asserts $0 < x + y + z < w'$, both inequalities of which were shown above. The second case is $\mathbf{S}_2\mathbf{b}' = (w, x', y, z)$. We assert that

$$w \le y \le z < x', \tag{3.11}$$

where we must show $x' > z$. Now $x' = 2(w+y+z)-x = (w+y+z)+(w+y+z-x) > w+y+z$ since property (P2) for $\mathbf{b}'$ gives $w + y + z > 2w + x + 2y \ge x$. Next $w + y \ge w + x \ge 0$ holds by property (P1) of $\mathbf{b}'$, so $w+y+z \geq z > 0$ whence $x' > z$, and (3.11) follows. Property (P1) for $\mathbf{S}_2\mathbf{b}'$ now asserts $w+y \geq 0$, which holds since $w+y+z \geq z$. Property (P2) for $\mathbf{S}_2\mathbf{b}'$ now asserts $0 < w+y+z < x'$, and both of these inequalities were verified above. Finally, the proof for $\mathbf{S}_3\mathbf{b}' = (w, x, y', z)$ is similar, where one shows $w \leq x \leq z < y'$. This completes the induction step, and the claim follows.

To continue the proof, we observe that for any non-root quadruple $\mathbf{b}'$ property (P2) implies that $L(\mathbf{b}) = (w+x+y)+z > 0$. Thus all quadruples in the packing have $L(\mathbf{b}) > 0$. We next show $\mathbf{b}$ is not reduced. Indeed property (P2) gives $\mathbf{S}_4\mathbf{b} = (w, x, y, z')$ with $0 < z' = 2(w+x+y) - z < z$ so that $|\mathbf{S}_4\mathbf{b}| < |\mathbf{b}|$. We conclude that the root quadruple is the unique reduced quadruple, which proves assertion (1).

To verify assertion (2), we must show that $|\mathbf{S}_j\mathbf{b}| \geq |\mathbf{b}|$ for $1 \leq j \leq 3$, while $|\mathbf{S}_4\mathbf{b}| < |\mathbf{b}|$. The last inequality is already done. Now $\mathbf{S}_3\mathbf{b} = (w, x, y', z)$ with $y' = 2(w+x+z) - y \geq 2(w+x) + 2(w+x+y) - y \geq y$ has $|\mathbf{S}_3\mathbf{b}| \geq |\mathbf{b}|$. The argument showing $|\mathbf{S}_j\mathbf{b}| \geq |\mathbf{b}|$ if $j = 1, 2$ is similar. This proves (2). $\square$

For $Q_{\mathcal{D}}(\mathbf{a}) = k$ with k fixed there are infinitely many distinct root quadruples. The next result determines various of their properties.

Theorem 3.3. *(1) Any root quadruple* $\mathbf{a} = (a, b, c, d)$ *with* $Q_{\mathcal{D}}(\mathbf{a}) = k$ *and* $a \leq b \leq c \leq d$ *and* $L(\mathbf{a}) > 0$ *satisfies*

$$0 \leq b \leq c \leq d. \tag{3.12}$$

In addition, if $k > 0$*, then*

$$a < 0, \tag{3.13}$$

and if $k \leq 0$ *then*

$$a \leq \sqrt{|k|}. \tag{3.14}$$

(2) For each pair of integers $(k, -n)$ *there are only finitely many root quadruples* $\mathbf{a}$ *with* $Q_{\mathcal{D}}(\mathbf{a}) = k$ *and* $a = -n$*, with the exception of those pairs* $(k, -n) = (4l^2, -l)$ *with* $l \geq 0$*. In each latter case there is an infinite family of root quadruples* $\{(-l, l, c, c) :\ c \geq \max(l, 1)\}$.

Proof. (1) The condition (3.3) implies $a + b \geq d - c \geq 0$, so that $b \geq 0$, and the ordering gives (3.12).

We view the Descartes equation $Q_{\mathcal{D}}(\mathbf{a}) = k$ as a quadratic equation in a, and solving it gives

$$a = b + c + d \pm \sqrt{4(bc + bd + cd) + k}$$

The ordering and (3.12) give $a \leq b \leq b + c + d$, which shows that the minus sign must be taken in the square root, and

$$a = b + c + d - \sqrt{4(bc + bd + cd) + k} \tag{3.15}$$

Using (3.3) we have

$$\begin{aligned}2(bc+bd+cd) &= (bc+d(b+c))+(bc+bd+cd)\\ &\geq (b^2+d(d-a)+(bc+bd+c^2)\\ &= b^2+c^2+d^2+[bc+d(b-a)],\end{aligned}$$

which gives

$$4(bc+bd+cd) \geq (b+c+d)^2+[bc+d(b-a)].$$

As a consequence (3.15) yields

$$a \leq (b+c+d)-\sqrt{(b+c+d)^2+k}. \tag{3.16}$$

For $k>0$ this immediately gives

$$a \leq b+c+d-\sqrt{(b+c+d)^2+1} < 0,$$

which is (3.13).

For $k \leq 0$ we must have $w := b+c+d \geq \sqrt{|k|}$ so that the square root above is real, whence (3.16) gives

$$a \leq \max_{w \geq \sqrt{|k|}} [w-\sqrt{w^2+k}] = \sqrt{|k|},$$

which is (3.14). Indeed the function $f(w) = w-\sqrt{w^2+k}$ on this domain is maximized at its left endpoint $w=\sqrt{|k|}$, since

$$f'(w) = 1-\frac{w}{\sqrt{w^2+k}} < 0 \quad \text{for} \quad w > \sqrt{|k|}.$$

Root quadruples with $a>0$ do occur for some negative k. For example $\mathbf{v}=(2,3,4,7)$ is a root quadruple for $k = Q_{\mathcal{D}}(\mathbf{v}) = -96$.

(2) We take the values $Q_{\mathcal{D}}(\mathbf{a}) = k$ and $a=-n$ as fixed. To show finiteness, it suffices to bound c above, in terms of k and n. The conditions $a+b+c \geq d$ and $0 \leq b \leq c \leq d$ yield $0 \leq d \leq 3c$, $0 \leq b \leq c$ and $-2c \leq a \leq c$.

We have $d = a+b+c-x$ for some $x \geq 0$, and substituting this in (3.15) gives

$$a = a+2b+2c-x-\sqrt{4(b+c)(a+b+c-x)+4bc+k}.$$

This simplifies to

$$0 = 2b+2c+x-\sqrt{(2b+2c-x)^2+(-x^2+4ab+4ac+4bc+k)},$$

so we must have

$$x^2 = 4ab+4ac+4bc+k. \tag{3.17}$$

Now $d \geq c+(a+b-x) \geq c$ gives

$$a+b \geq x \geq 0.$$

Squaring the left inequality and rearranging gives $a^2 + 2ab + b^2 \geq x^2 = 4ab + 4ac + 4bc + k$, which yields

$$(a - b)^2 \geq 4(a + b)c + k \tag{3.18}$$

We have $c \geq b \geq 0$, and holding a and k fixed and letting b grow, one sees that the right side grows at least as fast as $4b^2$ while the left side grows like b^2, so one concludes that b is bounded above. One can now check that (3.18) implies

$$b \leq 3|a| + |k|.$$

This bound shows that the left side of (3.18) is bounded above, yielding

$$4(a + b)c \leq (4|a|)^2 + |k|.$$

We have $a + b \geq x \geq 0$, and whenever $a + b > 0$ then this inequality bounds c above, with $c \leq (|a|)^2 + |k|$, as was required.

There remains the case $a+b = x = 0$. Letting $a = -l$, we have $b = l$ and $d = a+b+c-x = c$. Thus we have the quadruple $\mathbf{a} = (-l, l, c, c)$ with the requirement $c \geq l \geq 0$. All of these are root quadruples except for $l = c = 0$ which gives the excluded value $(0, 0, 0, 0)$. Here $Q(\mathbf{a}) = 4l^2$, and for each $l \geq 0$ we obtain the given infinite family of root quadruples. □

Remark. For $k \geq 0$ the upper bound $a \leq \sqrt{|k|}$ in Theorem 3.3(1) is not sharp in general. For the case $k = -4$ of spherical root quadruples, this upper bound gives $a \leq 2$, but all spherical root quadruples satisfy the stronger bound $a \leq 0$. To see this, observe that $a = 2$ can only hold with $b + c + d = 2$ in (3.16), and since $b \leq c \leq d$ are nonnegative integers we must have $b \leq 0$ so $a \leq 0$, a contradiction. Similarly $a = 1$ gives $1 \leq b + c + d \leq \frac{5}{2}$ in (3.16), with the same contradiction. The case $k = 0$ does occur with the spherical root quadruple $(0, 1, 1, 2)$ pictured in Figure 2.

Remark. We conclude this subsection by showing that the notions of reduction algorithm and root quadruple used here agree for $k = 0$ with those used in part I [8]. The part I reduction procedure applied only to quadruples with $L(\mathbf{a}) = a + b + c + d > 0$, and tried to greedily decrease the invariant $L(\mathbf{a}) = a + b + c + d$ at each step, halting if this could not be done. Although the general reduction algorithm uses a different reduction rule, in this special case it takes the identical series of steps as the reduction algorithm in part I, so defines the identical notion of root quadruple. Indeed, suppose $k = 0$ and $L(\mathbf{a}) > 0$. We will need a result from Theorem 3.4(3) below, which shows that there are no exceptional quadruples for $k = 0$. so the general reduction algorithm always halts at a root quadruple. Comparing Theorem 3.2(2) with Lemma 3.1 (iii) of [5] shows all the steps are identical. Namely, after reordering the curvatures in increasing order, both algorithms always apply $\mathbf{S}_4$. Finally we check that the halting rules coincide, i.e., that $|\mathbf{a}|$ is minimized where $L(\mathbf{a})$ is minimized. For both algorithms the minimal element in a root quadruple is non-positive, with the other three elements nonnegative (see Theorem 3.3(1) and [5, Lemma 3.1(3)]). Furthermore the first non-positive element encountered in either algorithm is never changed subsequently. Since the two algorithms coincide up to this point, and afterwards the two invariants remain in the fixed relation $|\mathbf{a}| = L(\mathbf{a}) + 2|a|$, for fixed $|a|$, their halting criteria coincide.

3.4. Exceptional Quadruples

We next consider orbits $\mathcal{A}[\mathbf{v}]$ that contain exceptional quadruples.

Theorem 3.4. *Let $\mathcal{A}[\mathbf{v}]$ be a nonzero integer orbit of the Apollonian group.*

(1) If $\mathcal{A}[\mathbf{v}]$ contains an exceptional quadruple, then it contains elements $\mathbf{v}_1, \mathbf{v}_2$ with $L(\mathbf{v}_1) > 0$ and $L(\mathbf{v}_2) < 0$.

(2) There can be more than one exceptional quadruple in $\mathcal{A}[\mathbf{v}]$.

(3) Exceptional quadruples can exist only for $k \geq 1$. For each $k \geq 1$ there are only finitely many exceptional quadruples with $Q_{\mathcal{D}}(\mathbf{a}) = k$, all with $H(\mathbf{a})^2 \leq 2k^2$.

Proof. (1) Let $\mathbf{a} = (a, b, c, d)$ be an exceptional quadruple in $\mathcal{A}[\mathbf{v}]$, with $a \leq b \leq c \leq d$. It suffices to treat the case $L(\mathbf{a}) = a + b + c + d \geq 0$, since we can otherwise apply the argument to $\mathbf{a}^*$.

Now Theorem 3.1 gives $a + b + c \leq 0$ and $d > 0$. Take $\mathbf{v}_1 = \mathbf{S}_1\mathbf{a} = (a', b, c, d)^T$ and $L(\mathbf{v}_1) = 3(a + b + c + d) - 4a > 0$ if $a < 0$, while if $a = 0$ then $a = b = c = 0$ and so $d > 0$ and $L(\mathbf{v}_1) > 0$ in this case. Next, take $\mathbf{v}_2 = \mathbf{S}_4\mathbf{a} = (a, b, c, d')$. Then $L(\mathbf{v}_2) = 3(a + b + c) - d < 0$ if $d > 0$, while if $d = 0$ then $a = b = c = d = 0$, a contradiction.

(2) The quadruple $\mathbf{a} = (-2, -3, -5, 12)$ with $|\mathbf{a}| = 22$ and $Q_{\mathcal{D}}(\mathbf{a}) = k = 360$ is an exceptional quadruple. Its neighbors are $(-2, -3, -5, -32), (-2, -3, 9, 12), (-2, 7, -5, 12)$, and $(6, -3, -5, 12)$. Now $(-2, -3, 9, 12)$ has a neighbor $\mathbf{b} = (-2, -3, 9, -4)$ which is also an exceptional quadruple.

(3) Let k be fixed. Let $\mathbf{a} = (a, b, c, d)$ be an exceptional quadruple ordered with $a \leq b \leq c \leq d$. By hypothesis

$$|\mathbf{S}_j(\mathbf{a})|^2 \geq |\mathbf{a}|^2 \quad \text{for} \quad 1 \leq j \leq 4. \tag{3.19}$$

We will show that necessarily $k \geq 1$ and that

$$H(\mathbf{a})^2 = a^2 + b^2 + c^2 + d^2 \leq 2k^2. \tag{3.20}$$

The minimality property (3.19) and the bound (3.20) are both invariant under reversing all signs, so by multiplying by -1 if necessary we may suppose that $L(\mathbf{a}) = a+b+c+d \geq 0$ without loss of generality. Since $(a, b, c, d) \neq (0, 0, 0, 0)$ we have $d > 0$. The condition $|\mathbf{S}_4\mathbf{a}|^2 \geq |\mathbf{a}|^2$ requires that either (i) $a+b+c \geq d$ or (ii) $a+b+c \leq 0$. By Theorem 3.1 exceptional quadruples correspond to case (ii), and those with $L(\mathbf{a}) > 0$ are characterized by the condition

$$d > 0 \geq a + b + c.$$

This condition necessarily implies $a \leq 0$. We consider three exhaustive cases. In each case, we change notation so that a, b, c are nonnegative; for example, if $a \leq 0 \leq b \leq c \leq d$, then we multiply a by -1 so that $\mathbf{a} = (-a, b, c, d)$.

Case 1. $-a \leq -b \leq -c \leq 0 < d$.

We have

$$\begin{aligned} k = Q_{\mathcal{D}}(\mathbf{a}) &= a^2 + b^2 + c^2 + d^2 - 2ab - 2ac - 2bc + 2ad + 2bd + 2cd \\ &= a^2 + b^2 + c^2 + d^2 + 2b(d - a) + 2a(d - c) + 2c(d - b) \\ &\geq a^2 + b^2 + c^2 + d^2 = H(\mathbf{a})^2. \end{aligned}$$

Since $\mathbf{a} \neq (0,0,0,0)$ we infer $k \geq 1$, and also (3.20) holds in this case.

Case 2. $-a \leq -b \leq 0 \leq c \leq d$.

We have

$$\begin{aligned} k = Q_{\mathcal{D}}(\mathbf{a}) &= a^2 + b^2 + c^2 + d^2 - 2ab + 2ac + 2bc + 2ad + 2bd - 2cd \\ &= (a-b)^2 + (c-d)^2 + 2ac + 2bc + 2ad + 2bd. \end{aligned}$$

Every term on the right is nonnegative, which implies $k \geq 0$. This gives the additional information (since $a \geq b$ and $d \geq c$) that

$$b \leq a \leq b + \sqrt{k} \quad \text{and} \quad c \leq d \leq c + \sqrt{k}.$$

as well as $ac, bc, ad, bd \leq \frac{k}{2}$. If either of a, b is nonzero then each of c, d is at most $\frac{k}{2}$. Similarly if either of c, d are nonzero then a, b are at most $\frac{k}{2}$. Now suppose both $a = 0$ and $b = 0$. Then $k = (c-d)^2$, and there is a reduction step that takes $(0,0,c,d) \mapsto (0,0,c,2c-d)$, whence $|2c - d| \geq d$, so that either $c = d$, or $c = 0$. The case $c = d$ gives a root quadruple, which is excluded, and in the remaining case $(0,0,0,d)$ we have $k = d^2 > 0$, and then $c, d \leq \sqrt{k}$, and in all these subcases

$$H(\mathbf{a})^2 = a^2 + b^2 + c^2 + d^2 \leq k^2$$

holds. The argument if $c = d = 0$ is similar. In all these cases we infer that $k > 0$, and that (3.20) holds.

Case 3. $-a \leq 0 \leq b \leq c \leq d$.

We have

$$\begin{aligned} k = Q_{\mathcal{D}}(\mathbf{a}) &= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac - 2bc + 2ad - 2bd - 2cd \\ &= (a+b)^2 + (c-d)^2 + 2(a-b)(c+d) \end{aligned} \tag{3.21}$$

In the exceptional case we must have $-a + b + c \leq 0$, so that $a \geq b + c$. This gives $a \geq b$, whence all terms on the right side of (3.21) are nonnegative, and one is strictly positive so $k > 0$. Also (3.21) gives $a, b \leq \sqrt{k}$, and using $a \geq b + c$ it also gives

$$k \geq (c-d)^2 + 2c(c+d) \geq 3c^2 + d^2.$$

It follows that

$$H(\mathbf{a})^2 = a^2 + b^2 + c^2 + d^2 \leq 2k \leq 2k^2,$$

and (3.20) holds. $\square$

On comparing Theorem 3.3 and Theorem 3.4 we see that: *An integer Apollonian group orbit* $\mathcal{A}[\mathbf{v}]$ *contains an exceptional quadruple if and only if it contains quadruples with* $L(\mathbf{x}_1) > 0$ *and* $L(\mathbf{x}_2) < 0$.

We conclude this section by determining the exceptional quadruples for spherical and hyperbolic Apollonian packings, i.e., for $k = \pm 4$. There are no exceptional quadruples for spherical packings ($k = -4$) by Theorem 3.4 (3), so we need only treat the hyperbolic case.

Theorem 3.5. *For $k = 4$ the exceptional orbits $\mathcal{A}[\mathbf{v}]$ are exactly those containing an element that has a zero coordinate. There are exactly two such orbits, one containing two exceptional quadruples $(0, 0, 0, \pm 2)$, and the other containing the exceptional quadruple $(-1, 0, 0, 1)$.*

Proof. By Theorem 3.2(3), there are finitely many exceptional quadruples for $k = 4$. From (3.20), we see that if (a, b, c, d) is an exceptional quadruple, then $a^2 + b^2 + c^2 + d^2 \leq 32$. A computer search shows that there are exactly 7 quadruples with $a \leq b \leq c \leq d$ satisfying this bound along with $a + b + c + d \geq 0$. They are

$$\{(-1, 0, 0, 1), (0, 0, 0, 2), (-1, 1, 1, 1), (-1, 1, 2, 2), (-1, 1, 3, 3), (0, 0, 1, 3), (0, 0, 2, 4)\}.$$

It is easy to check that the first five are all reduced while the last two are not. Furthermore, the third, fourth and fifth quadruples are root quadruples, so may be discarded. Thus only $(-1, 0, 0, 1), (0, 0, 0, 2)$ remain, and these are exceptional. The orbit of $(0, 0, 0, 2)$ also contains the reduced quadruple $(-2, 0, 0, 0)$; see Figure 4. The packing with exceptional quadruple $(-1, 0, 0, 1)$ is shown in Figure 5.

If fact, if $\mathcal{A}[\mathbf{v}]$ does contain a circle of curvature 0, then it necessarily contains an exceptional quadruple. Setting $a = 0$, the orbit contains a integer quadruple $(0, b, c, d)$ with

$$F(b, c, d) := b^2 + c^2 + d^2 - 2(bc + bd + cd) = 4. \tag{3.22}$$

Without loss of generality, assume that the elements are ordered $|b| \leq |c| \leq |d|$ and that $b \leq 0$. We claim that necessarily $b = 0$.

Case 1. $b \leq 0$, $c > 0$, $d > 0$.
Now

$$\begin{aligned} f(b, c, d) &= b^2 + c^2 + d^2 + 2|bc| + 2|bd| - 2|cd| \\ &= b^2 + (c - d)^2 + 2|bc| + 2|bd| \end{aligned}$$

If $|b| \geq 1$, then $|c|, |d| \geq 1$ also and $F(b, c, d) \geq 1^2 + 0^2 + 2 + 2 = 5$ which gives us a contradiction. So we must have $b = 0$.

Case 2. $b \leq 0$, $c \leq 0$, $d > 0$.
Here $|b| \leq |c| \leq |d|$ gives $|bd| > |bc|$, so

$$\begin{aligned} F(b, c, d) &= b^2 + c^2 + d^2 + 2|bd| + 2|cd| - 2|bc| \\ &\geq b^2 + c^2 + d^2 + 2|cd| \end{aligned}$$

If $|b| \geq 1$, then so are $|c|$ and $|d|$, so $F(b, c, d) \geq 5$, a contradiction. Thus $b = 0$.

Case 3. $b, c, d \leq 0$.
First, apply the reduction algorithm to $(0, b, c, d)$. Notice that the first coordinate always stays 0 because $\mathbf{S}_1$ can never make it any smaller. The algorithm halts when $\mathbf{S}_4$ can no longer reduce $|b| + |c| + |d|$, that is, when $|d'| \geq |d|$. Combining this with $d + d' = 2(b + c)$, we get $|d| > |b + c|$. Therefore

$$b^2 + c^2 + d^2 \leq b^2 + c^2 + (b + c)^2 = 2b^2 + 2bc + 2c^2.$$

Since $bc, bd, cd \geq 0$, we have $bc + bd + cd \geq |b|^2 + |bc| + |c|^2$. Thus,

$$F(b, c, d) = b^2 + c^2 + d^2 - 2(bc + bd + bd) \leq 0$$

and hence cannot satisfy (3.22), a contradiction. So there are no solutions in this case.

The claim $b = 0$ is proved, so any quadruple with one element zero is contained in the same packing as some $(0, 0, c, d)$. Now solve $2(c^2 + d^2) = (c + d)^2 + 4$ and we see that we must have $(0, 0, c, d) = (0, 0, n, n + 2)$ for some $n \in \mathbb{Z}$. Reduction by the Apollonian group changes $(0, 0, n, n + 2) \to (0, 0, n, n - 2)$, and therefore there are only two packings containing a curvature zero circle, those are the packing containing $(0, 0, 0, 2)$ and the packing containing $(-1, 0, 0, 1)$. $\square$

To understand the geometric nature of the exceptional hyperbolic packings, we identify the real hyperbolic plane $\mathbb{H}$ with the interior of the unit disk, with ideal boundary the circle of radius 1 centered at the origin. Recall that the curvature of a hyperbolic circle is given by $\coth r$, where r is the hyperbolic radius of the circle. For circles contained in the real hyperbolic plane, r is positive and

$$\coth r = \frac{e^r + e^{-r}}{e^r - e^{-r}} \geq 1$$

with equality if and only if $r = \infty$. However we will allow hyperbolic Apollonian circle packings which lie in the entire plane. We have a second copy of the hyperbolic plane which is the exterior of the closed unit disk, and we use the convention that (positively oriented) circles entirely contained in the exterior region have the sign of their curvature reversed, in which case $\coth r$ ranges from -1 to $-\infty$. Euclidean circles that intersect the ideal boundary (unit circle) are treated as hyperbolic circles of pure imaginary radius, and these cover the remaining range $-1 \leq \coth r \leq 1$. A hyperbolic circle of curvature 0 is a hyperbolic geodesic, which corresponds to a Euclidean circle which intersects the ideal boundary (unit circle) at right angles, and a hyperbolic circle of curvature 1 is a horocycle, i.e., a circle tangent to the ideal boundary.

The two exceptional integral hyperbolic packings are pictured in Figures 4 and 5, respectively. The pictures are not drawn to the same scale; in each one the dotted circle represents the ideal boundary of hyperbolic space, and is not a circle belonging to the packing. The hyperbolic packing $(0, 0, 0, 2)$ contains exactly three circles assigned hyperbolic curvature zero, while the hyperbolic packing $(-1, 0, 0, 1)$ contains infinitely many circles assigned hyperbolic curvature zero.

4. Distribution of Integer Root Quadruples

In this section we restrict to the case $k = 4m$. We enumerate root quadruples $\mathbf{v} = (w, x, y, z)$, and without essential loss of generality we only consider such quadruples with $L(\mathbf{v}) = w + x + y + z > 0$. The root quadruples are classified according to the size of their minimal element, say $w = -n$, where we suppose $n > 0$. In the spherical and hyperbolic cases $k = \pm 4$ this corresponds geometrically to characterizing the distinct integral Apollonian packings that have an outer enclosing circle of signed curvature $-n$.

Definition 4.1. *Let $N_{root}(k; -n)$ count the number of integer root quadruples $\mathbf{v} = (w, x, y, z)$ to $Q_{\mathcal{D}}(\mathbf{v}) = k$ with $L(\mathbf{v}) = w + x + y + z > 0$ and $w = -n$.*

In [8, Sec. 4] the number $N_{\text{root}}(0, -n)$ of Euclidean root quadruples was interpreted as a class number for binary quadratic forms under a $GL(2, \mathbb{Z})$ action. This interpretation was discovered starting from an exact formula for the number of (Euclidean) root quadruples having

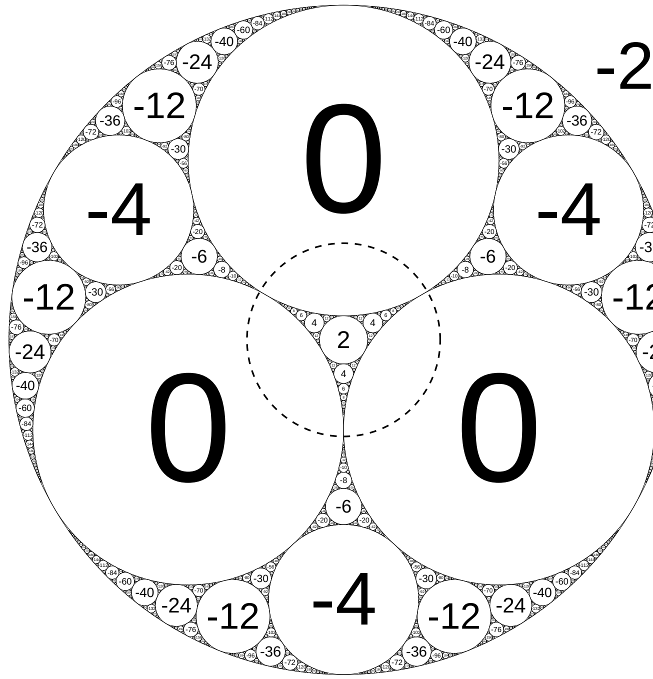

Figure 4: The exceptional hyperbolic Apollonian circle packing (0,0,0,2)

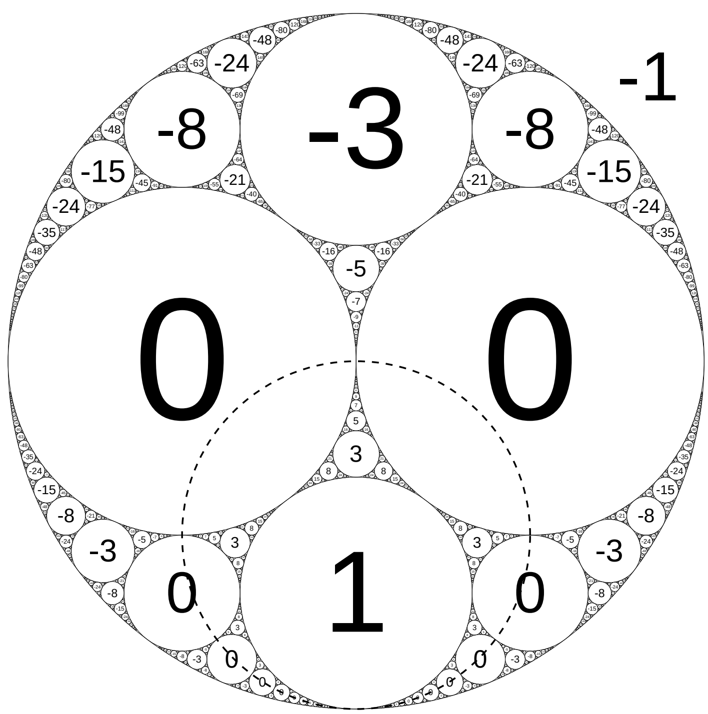

Figure 5: The exceptional hyperbolic Apollonian circle packing (-1,0,0,1)

smallest element $-n$, originally conjectured by C. L. Mallows and S. Northshield, and independently proved by Northshield [15]. Below we show that a similar class number interpretation holds generally for the case $Q_{\mathcal{D}}(\mathbf{a}) = 4m$.

We write a binary quadratic form of even discriminant $\Delta = 4B^2 - 4AC$ as

$$[A, B, C] := AT^2 + 2BTU + CU^2. \tag{4.1}$$

Here we follow the classical notation of Mathews [13], to maintain consistency with the notation of [8]. (The convention of Buell [2] writes this form as $[A, 2B, C]$, and allows odd middle coefficients.) A form is *definite* if $\Delta < 0$, and it is then *positive* if it only represents positive values (and zero). Positive definite forms are those definite forms with both A and C positive, which follows from $-4AC \le 4(B^2 - AC) < 0$.

The standard action on binary forms is an $SL(2,\mathbb{Z})$-action. A *reduced form* is one that satisfies

$$2|B| \le A \le C. \tag{4.2}$$

Every positive definite form is equivalent (under the $SL(2,\mathbb{Z})$-action) to at least one *reduced form*. All reduced forms are $SL(2,\mathbb{Z})$-inequivalent except for $[A, B, A] \equiv [A, -B, A]$ and $[A, A, C] \equiv [A, -A, C]$.

There is also a $GL(2,\mathbb{Z})$-action on the space of binary quadratic forms. It defines an action of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on binary forms, sending $[A, B, C]$ to $[A, -B, C]$. We define the $GL(2,\mathbb{Z})$-reduced forms to be the subset of $SL(2,\mathbb{Z})$-reduced forms satisfying

$$0 \le 2B \le A \le C. \tag{4.3}$$

Then every positive definite form is $GL(2,\mathbb{Z})$-equivalent to a unique $GL(2,\mathbb{Z})$-reduced form.

A form is *primitive* if $\gcd(A, 2B, C) = 1$ and is *imprimitive* otherwise. Define $\widetilde{h^{\pm}}(\Delta)$ to be the number of (primitive or imprimitive) $GL(2,\mathbb{Z})$-reduced forms of discriminant Δ. Let $\widetilde{h}(\Delta)$ be the number of (primitive or imprimitive) $SL(2,\mathbb{Z})$-equivalence classes of forms of discriminant Δ. Since each $GL(2,\mathbb{Z})$-reduced form $[A, B, C]$ is equivalent to the $SL(2,\mathbb{Z})$-reduced form $[A, -B, C]$, there holds

$$\widetilde{h^{\pm}}(\Delta) = \frac{1}{2}(\widetilde{h}(\Delta) + \widetilde{a}(\Delta)), \tag{4.4}$$

in which $\widetilde{a}(\Delta)$ counts the number of (primitive or imprimitive) ambiguous classes. Here an *ambiguous class* is an $SL(2,\mathbb{Z})$-equivalence class containing an *ambiguous form*, which is a (primitive or imprimitive) reduced form for which $2B$ divides A, i.e., reduced forms of shape $[A, 0, C]$ or $[2B, B, C]$, cf. Mathews [13, Art. 159].

The usual class number $h(\Delta)$ is defined as the number of $SL(2,\mathbb{Z})$-equivalence classes of primitive forms. Similarly we let $a(\Delta)$ count the number of primitive ambiguous classes. If $[A, B, C]$ is a primitive form of discriminant Δ, then for $\ell > 1$, $[\ell A, \ell B, \ell C]$ is an (imprimitive) form of discriminant $\ell^2\Delta$. We have

$$\widetilde{h}(\Delta) = \sum_{\ell^2 | \Delta} h\left(\frac{\Delta}{\ell^2}\right), \tag{4.5}$$

where we use the convention that $h(n) = 0$ if n is not a valid discriminant. In the same vein we have

$$\widetilde{a}(\Delta) = \sum_{\ell^2|\Delta} a\left(\frac{\Delta}{\ell^2}\right). \tag{4.6}$$

For $\Delta = -4m < 0$ one has $a(\Delta) = 2^{t+1}$, where t is the number of distinct prime divisors of m; see Mathews [13, Art. 156].

The following result gives a class number criterion counting root quadruples for an arbitrary Descartes equation $Q_{\mathcal{D}} = 4m$. This result applies even when the associated discriminant Δ is non-negative, provided the definition of reduced form is given by (4.2) (resp. (4.3)), though this no longer corresponds nicely to $SL(2,\mathbb{Z})$-equivalence (resp. $GL(2,\mathbb{Z})$-equivalence).

Theorem 4.1. *Let $n > 0$ be fixed.*

(1) The ordered root quadruples $\mathbf{v} = (w, x, y, z)$ with $Q_{\mathcal{D}}(\mathbf{v}) = 4m$, for fixed m, having smallest element $w = -n$ are in one-to-one correspondence with reduced integral binary quadratic forms $[A, B, C] = AX^2 + 2BXY + CY^2$ (primitive or imprimitive) having discriminant $\Delta = -4n^2 + 4m$ and a non-negative middle coefficient. If $\mathbf{v} = (w, x, y, z)$ with $w = -n \leq x \leq y \leq z$ then the correspondence is given by

$$[A, B, C] = \left[-n + x, \frac{1}{2}(-n + x + y - z), -n + y\right]. \tag{4.7}$$

(2) When $\Delta = -4n^2 + 4m < 0$, the number of root quadruples to $Q_{\mathcal{D}} = 4m$ with least element $-n$ is given by the $GL(2,\mathbb{Z})$-class number

$$N_{root}(4m; -n) = \widetilde{h^{\pm}}(-4(n^2 - m)). \tag{4.8}$$

Proof. Recall from Theorem 3.1 that an ordered integer quadruple $\mathbf{v} = (w, x, y, z)$ with $w \leq x \leq y \leq z$ and $L(\mathbf{v}) > 0$ is a root quadruple if and only if

$$w + x + y \geq z \geq 0, \tag{4.9}$$

and Theorem 3.3 shows that root quadruples always satisfy $0 \leq x \leq y \leq z$. We now write $w = -n$ to suggest that we are mainly concerned with negative integers. However, for negative m small positive values of w are permitted, which necessarily satisfy $w \leq \sqrt{|m|}$, by Theorem 3.3(1).

The integer solutions $Q_{\mathcal{D}}(w, x, y, z) = 4m$ with $w = -n$ are in one-to-one correspondence with integer representations of $n^2 - m$ by the ternary quadratic form

$$Q_T(X, Y, Z) := XY + XZ + YZ,$$

where

$$(X, Y, Z) := \left(\frac{1}{2}(w + x + y - z), \frac{1}{2}(w + x - y + z), \frac{1}{2}(w - x + y + z)\right).$$

The congruence condition $w + x + y + z \equiv 0 \pmod 2$ for integer quadruples implies that $X, Y, Z \in \mathbb{Z}$. This map is a bijection: given an integer solution to $Q_T(X, Y, Z) = n^2 - m$, we write

$$(w, x, y, z) := (-n, n + X + Y, n + X + Z, n + Y + Z)$$

and an algebraic calculation shows that $Q_{\mathcal{D}}(w, x, y, z) = 4m$.

The root quadruple conditions (4.9) above translate to the inequalities

$$0 \le X \le Y \le Z. \tag{4.10}$$

Next, for any integer M, the integer solutions $Q_T(X, Y, Z) = M$ are in bijection with integer representations of $-4M$ by the determinant ternary quadratic form

$$Q_\Delta(A, \tilde{B}, C) := \tilde{B}^2 - 4AC$$

A solution (X, Y, Z) gives a solution $Q_\Delta(A, \tilde{B}, C) = -4M$ under

$$(A, \tilde{B}, C) := (X + Y, 2X, X + Z),$$

and the inverse map is

$$(X, Y, Z) := (\tilde{B}/2, A - \tilde{B}, C - \tilde{B}).$$

Note that since $-4M = \tilde{B}^2 - 4AC$ is even, then $\tilde{B}$ must also be even, so $B = \tilde{B}/2$ must be an integer.

One sees easily that the inequalities (4.10) are equivalent to the inequalities

$$0 \le 2B \le A \le C. \tag{4.11}$$

Finally, we recognize that the conditions (4.11) give a complete set of equivalence classes of integral binary quadratic forms of fixed discriminant $-4M$ under the action of $GL(2, \mathbb{Z})$.

Therefore, taking $M = n^2 - m$, these two steps associate to any (ordered) integer quadruple $(-n, x, y, z)$ the binary quadratic form

$$[A, B, C] = \left[-n + x, \frac{1}{2}(-n + x + y - z), -n + y\right]$$

of discriminant $D := -4(n^2 - m)$. In order for the form to be positive definite, we must have $n^2 > m$. This imposes no constraint on n if $m < 0$, while if $m \ge 0$ we must have $n > \sqrt{|m|}$. This form is $GL(2, \mathbb{Z})$-reduced if and only if $(-n, x, y, z)$ is a root quadruple.

Conversely, given a form of discriminant D, we construct a Descartes quadruple

$$(w, x, y, z) := (-n, n + A, n + C, n + A + C - 2B)$$

which is a root quadruple if and only if $[A, B, C]$ is $GL(2, \mathbb{Z})$-reduced. □

Theorem 4.1 counts the number of root quadruples $N_{\text{root}}(4m; -n)$ with $L(\mathbf{v}) > 0$ having a fixed smallest element $w = -n$ in terms of a class number, provided $n^2 > m$. This condition

n	$N^S(-n)$	$N^H(-n)$	n	$N^S(-n)$	$N^H(-n)$	n	$N^S(-n)$	$N^H(-n)$
1	1	$+\infty$	11	6	12	21	6	24
2	2	2	12	6	12	22	12	12
3	2	3	13	8	12	23	16	22
4	3	4	14	6	12	24	5	24
5	4	6	15	5	19	25	19	24
6	2	6	16	9	16	26	16	25
7	5	8	17	12	18	27	8	24
8	6	8	18	10	10	28	10	28
9	3	11	19	10	24	29	14	24
10	8	9	20	11	20	30	14	30

Table 1: $N_{\text{root}}(\pm 4, -n)$ for small n.

is always satisfied when $m < 0$. For the case $m = 0$ there is one excluded value $n = 0$ corresponding to multiples of the root quadruple $(0, 0, 1, 1)$, which falls under Theorem 3.3(2). For positive m and for the remaining values $-\sqrt{|m|} \le w < 0$ allowed by Theorem 3.3, the combinatorial correspondence with certain binary quadratic forms of discriminant $\Delta = -4(n^2 - m) \ge 0$ still applies, but the notion of "reduced form" in (4.2) is not that that normally used for indefinite binary quadratic forms, and $N_{\text{root}}(4m, -n)$ is not interpretable as a class number. For hyperbolic Descartes quadruples ($m = 1$), there is one allowable value $n = -1$ not interpretable as a class number. Indeed there are infinitely many root quadruples when $n = -1$ by Theorem 3.3(2), since $(k, n) = (4m^2, -m)$ with $m = 1$.

Table 1 presents small values of $N^S_{\text{root}}(-n) := N_{\text{root}}(-4, -n)$ and $N^H_{\text{root}}(-n) := N_{\text{root}}(4, -n)$. All hyperbolic root quadruples have $-n \le -1$, according to Theorem 3.3(1), and all spherical root quadruples have $n \le 0$, as shown in the proof of Theorem 3.3(1). The value $n = 0$ is not shown in the table; we have $N^S_{\text{root}}(0) = 1$, given by the spherical root quadruple $(0, 1, 1, 2)$.

The next result uses bounds for class numbers to derive an upper bound for the number of root quadruples.

Theorem 4.2. *For all integers m the number $N_{root}(4m; -n)$ of ordered integer root quadruples (a, b, c, d) with $Q_{\mathcal{D}}(\mathbf{a}) = 4m$ and $a = -n$ satisfies, uniformly for $n > \sqrt{|m|}$,*

$$N_{root}(4m; -n) = O\left(n(\log n)(\log\log n)^2\right).$$

The implied O-constant is independent of m.

Proof. Since $\widetilde{h}^{\pm}(\Delta) \le \widetilde{h}(\Delta)$ we have

$$N_{\text{root}}(4m; -n) \le \widetilde{h}(-4(n^2 - m)) = \sum_{\ell^2 \,\mid\, -4(n^2-m)} h\left(\frac{-4(n^2 - m)}{\ell^2}\right).$$

in which $h(D)$ denotes the usual class number for primitive forms under $SL(2, \mathbb{Z})$-equivalence. Dirichlet's class number formula gives, for primitive negative discriminants D, that

$$h(D) = \frac{w(D)\sqrt{|D|}}{2\pi} L(1, \chi),$$

where χ denotes the (primitive or imprimitive) character attached to the quadratic field $\mathbb{Q}(\sqrt{D})$, and $w(D)$ denotes the number of roots of unity in this field, which is at most six. This formula is valid for class numbers of non-maximal orders as well, where the L-function now uses an imprimitive character. For primitive characters we have the bound

$$L(1,\chi) = O(\log|D|);$$

see Davenport [4, Chap. 14], and Ramaré [16] for precise estimates. For imprimitive characters, we must correct by a finite Euler factor $\prod_{p\in S}(1-\chi(p)\frac{1}{p})$ involving some subset S of primes dividing the discriminant. As shown in [8, Theorem 4.4], this is bounded above by $\prod_{p\in S}(1+\frac{1}{p})$, and can lead to an extra factor of size at most $O(\log\log|D|)$. Writing $D = D_0S^2$ with D_0 a fundamental discriminant, we have

$$\begin{aligned}\widetilde{h}(D) &= O\left(\sqrt{|D_0|}\log|D_0|\sum_{d\mid S} d\prod_{p\mid d}\left(1+\frac{1}{p}\right)\right)\\ &= O\left(\sqrt{|D_0|S^2}\log|D_0|(\log\log S)^2\right). \qquad (4.12)\end{aligned}$$

This gives

$$\widetilde{h}(D) = O\left(\sqrt{|D|}\log|D|(\log\log|D|)^2\right),$$

valid uniformly for all discriminants $D \le -4$. Here we used the estimate $\sum_{d\mid S} d \le S\log\log S$. The result follows on taking $D = -4(n^2-m)$. The O-constant can be taken independent of m since $n^2 - m \le 2n^2$ under the stated hypotheses. □

Remarks. (1) The upper bound of Theorem 4.2 can be strengthened when $m = 0$. In that case $D = -4n^2$, and one obtains the stronger upper bound $O\left(n(\log\log n)^2\right)$ using (4.12) with $D_0 = -4$ and $S = n^2$. This improved upper bound also follows from [8, Theorem 4.2]. The latter result counts primitive root quadruples only and gets a bound $O(n(\log\log n))$. Theorem 4.2 counts primitive and imprimitive root quadruples (with primitivity defined by the auxiliary binary quadratic form (4.7)). Summing over the imprimitive forms produces an extra factor of $\log\log n$.

(2) In [8] a lower bound $\Omega\left(\frac{n}{\log\log n}\right)$ was obtained for the number of primitive Euclidean root quadruples, which implies

$$N_{\text{root}}(0,-n) = \Omega\left(\frac{n}{\log\log n}\right).$$

We are unable to obtain a lower bound for of equivalent strength for general $k = 4m$, because it appears to require obtaining good lower bounds for class numbers of general imaginary quadratic fields, which is a difficult problem. Using the Brauer-Siegel theorem (see [11, pg. 328]) one can obtain for fixed m and any $\epsilon > 0$ a lower bound

$$N_{\text{root}}(4m,-n) \ge n^{1-\epsilon}$$

valid for all $n \geq n_0(m, \epsilon)$, in which the constant $n_0(m, \epsilon)$ is not effectively computable.

We end this section by raising some questions concerning the behavior of the total number of root quadruples with $Q_{\mathcal{D}}(\mathbf{v}) = k$ having minimal element below a given bound. We define the summatory function

$$S(k, T) := \sum_{n=\lfloor \sqrt{|k|} \rfloor + 1}^{T} N_{\text{root}}(k, -n),$$

Theorem 4.2 gives for nonzero $k = 4m$ that

$$S(4m, T) = \sum_{n=\lfloor \sqrt{|k|} \rfloor + 1}^{T} \widetilde{h}^{\pm}(4(n^2 + m)). \tag{4.13}$$

One expects these sums of $GL(2, \mathbb{Z})$-class numbers over a quadratic sequence to grow roughly like $T^{3/2}$. Can the leading term in their asymptotics be determined? The problem can be reduced to considering similar sums of $SL(2, \mathbb{Z})$-class numbers using (4.4) to obtain

$$S(4m, T) = \frac{1}{2} \sum_{n=\lfloor \sqrt{|k|} \rfloor + 1}^{T} \widetilde{h}(-4(n^2 + m)) + O\left(T \cdot 2^{\frac{\log T}{\log \log T}}\right),$$

where we made use of an upper bound for the number of ambiguous forms, as follows. If t denotes the number of distinct prime divisors of $\Delta < 0$ then (4.6) yields

$$\tilde{a}(\Delta) \leq d(\Delta) 2^{t+1} \leq 2^{O(\frac{\log \Delta}{\log \log \Delta})}.$$

One can also ask whether $\frac{S(4,T)}{S(-4,T)}$ tends to a limit as $T \to \infty$, and if so, compare the limiting value to that in (2.7). One expects that there are asymptotically fewer spherical root quadruples than hyperbolic root quadruples. However, it is not true that $N_{\text{root}}(-4, n) \leq N_{\text{root}}(4, n)$ always. For example, $N_{\text{root}}(-4, 32) = 23 > 20 = N_{\text{root}}(4, 32)$.

5. Integers Represented by a Packing: Congruence Conditions

In this section we restrict to the cases $k = \pm 4$ corresponding to spherical and hyperbolic Apollonian packings. We show there are congruence restrictions on the allowed "curvatures" modulo 12 of circles in integral spherical and hyperbolic Apollonian circle packings.

Theorem 5.1. *In any integral spherical Apollonian circle packing, the "curvatures" of the circles in the packing omit exactly three congruence classes modulo* 12.

Proof. It suffices to classify the unordered Descartes quadruples in all integral packings. We claim that these fall into one of two eighteen-element orbits mod 12, which are $\mathcal{O}$ and $\mathcal{O} + 6$ (mod 12), where

$$\begin{aligned}\mathcal{O} = \{&(2,1,1,0), (5,2,1,0), (5,4,2,1), (6,5,4,1), (6,5,5,4), (8,2,1,1),\\ &(8,6,1,1), (8,6,5,1), (9,4,2,1), (9,5,4,2), (9,8,2,1), (10,5,1,0),\\ &(10,5,5,0), (10,5,5,4), (10,8,5,1), (10,9,5,4), (10,9,8,1), (10,9,8,5)\},\end{aligned}$$

and the notation $\mathcal{O}+6 \pmod{12}$ means to add 6 to each coordinate of each element of $\mathcal{O}$ and reduce modulo 12.

The theorem follows from the claim, since the curvatures in $\mathcal{O}$ omit the classes $3, 7$ and $11 \pmod{12}$, while $\mathcal{O}+6$ omits $1, 5$ and $9 \pmod{12}$.

To check the claim, it is easy to verify that there are 212 possible solutions (without respect to order) of the spherical Descartes equation (1.4) modulo 12. These lie in 19 orbits under the action of the Apollonian group. However, not all of the solutions modulo 12 come from integer solutions.

If $\mathbf{a} = (w, x, y, z)$ is the reduction modulo 12 of a solution to the spherical Descartes equation, then there must exist a, b, c, d such that

$$2((12a+w)^2 + (12b+x)^2 + (12c+y)^2 + (12d+z)^2) = \\ (12(a+b+c+d) + (w+x+y+z))^2 - 4 \quad (5.1)$$

If we reduce this equation modulo 24, we get

$$2(w^2+x^2+y^2+z^2) \equiv (w+x+y+z)^2 - 4 \pmod{24},$$

and we see that $\mathbf{a}$ must also be a Descartes quadruple mod 24. This eliminates 108 of the 212 possibilities and 11 of the 18 orbits. Furthermore, if we reduce (5.1) modulo 48, we get the condition that

$$2(w^2+x^2+y^2+z^2) + 24(a+b+c+d)(w+x+y+z) \\ \equiv (w+x+y+z)^2 - 4 \pmod{48}.$$

But since $w+x+y+z \equiv 0 \pmod{2}$, this is

$$2(w^2+x^2+y^2+z^2) \equiv (w+x+y+z)^2 - 4 \pmod{48}. \quad (5.2)$$

Thus (w, x, y, z) must also be a quadruple modulo 48, ruling out 68 of the 104 remaining quadruples. Finally, a short computer search turns up integer spherical quadruples in all 36 classes. ☐

Theorem 5.2. *In any integral hyperbolic Apollonian circle packing, the "curvatures" of the circles in the packing omit at least three congruence classes modulo* 12*.*

Proof. We classify the (unordered) Descartes quadruples in an integral hyperbolic packing, showing that they fall in one of 81 equivalence classes modulo 12. We claim that the integer quadruples lie in 15 orbits under the action of the Apollonian group, given by $\mathcal{O}_3, \mathcal{O}_3+3, \mathcal{O}_3-3$, $-\mathcal{O}_3, -\mathcal{O}_3+3, -\mathcal{O}_3-3$, $\mathcal{O}_4, \mathcal{O}_4+6$, $-\mathcal{O}_4, -\mathcal{O}_4+6$, $\mathcal{O}_7, \mathcal{O}_7+3, \mathcal{O}_7+9$ and $\mathcal{O}_{13}, \mathcal{O}_{13}+6$, where

$$\mathcal{O}_3 = \{(2,4,8,8), (2,8,8,8), (8,8,8,10)\}$$
$$\mathcal{O}_4 = \{(1,2,2,11), (2,2,5,11), (2,2,5,7), (2,5,10,11)\}$$
$$\mathcal{O}_7 = \{(0,0,0,10), (0,0,0,2), (0,0,2,4), (0,0,4,6), (0,0,6,8), (0,0,8,10), (0,4,6,8)\}$$
$$\mathcal{O}_{13} = \{(0,0,1,11), (0,0,1,3), (0,0,3,5), (0,0,5,7), (0,0,7,9), (0,0,9,11), (0,1,3,8), \\ (0,3,4,5), (0,3,4,9), (0,3,8,9), (0,4,9,11), (0,7,8,9), (3,4,8,9)\}.$$

It is easy to check that each of these orbits omits at least three residue classes modulo 12; for example $\mathcal{O}_3$ omits the eight classes $0, 1, 3, 5, 6, 7, 9$, and $11 \pmod{12}$, so the theorem follows from the claim.

To prove the claim, we first calculate there are 278 solutions to the hyperbolic Descartes equation mod 12, and then check how many of these lift to global integer solutions of the hyperbolic Descartes equation. Of these solutions, 184 can be eliminated by consideration modulo 24 and 48 as in the spherical case. However, 13 quadruples not on the above list work modulo 48. These 13 quadruples lie in three orbits, namely

$$\begin{aligned} \mathcal{O}_3 + 6 &= \{(2,2,2,4),(2,2,2,8),(2,2,8,10)\},\\ \mathcal{O}_3 + 8 &= \{(2,4,10,10),(4,10,10,10),(8,10,10,10)\},\\ \mathcal{O}_7 + 6 &= \{(0,2,6,10),(0,2,6,6),(0,6,6,10),(2,4,6,6),(4,6,6,6),(6,6,6,8),(6,6,8,10)\}. \end{aligned} \tag{5.3}$$

To eliminate these quadruples, we reduce the hyperbolic Descartes equation modulo 96, which yields

$$\begin{aligned} &2((12a+w)^2+(12b+x)^2+(12c+y)^2+(12d+z)^2) \equiv \\ &\qquad (12(a+b+c+d)+(w+x+y+z))^2+4 \pmod{96}. \end{aligned} \tag{5.4}$$

After substituting $(w, x, y, z) = (2, 2, 2, 4)$, $(2, 4, 10, 10)$, or $(0, 2, 6, 10)$ into (5.4) we get

$$48\alpha^2 + 48\alpha + 48 \equiv 0 \pmod{96}$$

where $\alpha = a + b + c + d$. This equation has no solutions as $\alpha^2 + \alpha + 1 \equiv 1 \pmod 2$ for any α. We have shown that one quadruple from each of the orbits (5.3) is impossible, thus ruling out the entire orbit.

Finally, a quick search shows that the other 15 orbits all occur as reductions of integer hyperbolic Apollonian quadruples. □

In [8], it is conjectured that in any integral Euclidean Apollonian packing, all sufficiently large integers not ruled out by congruence conditions modulo 24 are represented. This was suggested by numerical data for a few specific packings. Table 2 presents integers below 100000 missing from the spherical Apollonian packing $(0, 1, 1, 2)$ grouped by residue classes modulo 24; we give data for residue classes $0, 1, 2, 5 \pmod{24}$ as representative. It is not completely clear from this data whether one should believe that only a finite number of integers will be missed in each class. For certain residue classes modulo 24 the exceptions seem to be rapidly thinning out. However for the class $0 \pmod{24}$ (and also 9, 18 and 21 (mod 24)) the data seems equivocal.

It seems very likely that there are no congruence restrictions on integers in spherical and hyperbolic circle packings for any modulus prime to 6. In [8, Theorem 6.2], it is shown that in any integral Euclidean Apollonian packing, the "curvatures" of the circles include representatives of every residue class modulo m for any modulus m that is relatively prime to 30. We expect a similar result to hold for spherical and hyperbolic packings as well.

$n \equiv 0 \pmod{24}$

24	48	144	168	264	384	504	528	720	792
864	960	984	1008	1104	1344	1392	1632	1728	1824
2208	2232	2352	2448	2592	2904	3144	3192	3384	3744
3984	4032	4104	4248	4464	4584	4944	5352	5424	5664
5784	6048	6384	6624	7272	7344	7464	7776	8064	8160
8664	8712	8808	8904	9264	9312	9984	10032	10224	10248
10368	10752	11064	11184	11304	11424	11592	11952	12648	12864
13272	13368	13584	13704	13824	14064	14160	14304	15144	15552
15624	15984	16704	16872	17712	17784	18192	18264	18768	19224
19704	19872	19944	20304	20424	20664	22104	22632	23304	23568
24744	24816	24984	25248	25944	26544	26904	28008	28224	28392
29376	29784	29976	30072	30744	32544	32904	33552	33744	33888
33936	33984	36024	36672	36984	37104	37464	37632	38424	39024
39096	39624	40104	40152	41784	41904	41952	42816	43272	43584
43848	43896	43944	44856	45144	45912	46752	46824	46872	47184
47784	49704	49992	50160	50448	50928	51312	51504	51552	52584
53424	54816	54864	55464	55728	55824	56736	56856	58344	58824
60528	60744	61872	62424	63120	64224	64344	64824	64944	65544
65664	66144	66744	66984	68304	68472	70344	71712	71856	72744
73944	76560	76584	77592	77664	78384	79584	80664	81264	81984
83784	84384	84624	85536	87744	88152	88344	92712	93024	93672
93864	94584	95184	96144	96216	97848	99264	99792	99864	99984

$n \equiv 1 \pmod{24}$

49	217	529	553	889	1441	1897	2737	3073	3337
4009	4417	4609	6049	7273	8449	9289	9889	10609	10921
11017	11257	11809	11929	12649	14449	15289	18529	18601	20209
23017	24577	27769	37009	39577	43489	45649	46129	47449	51049
52369	54313	54529	55369	56809	66289	66889	73249	80569	83329
91129	91729	92329							

$n \equiv 2 \pmod{24}$

434	506	1034	3626	7706	8786	12674	19634	25154	27554
28034	29714	41354	83426	97850					

$n \equiv 5 \pmod{24}$

77	749	869	893	1301	2189	2261	2429	2573	3317
3509	5789	6077	9437	16469	17789	19589	22589	22709	44549
56909	72869								

Table 2: Missing curvatures below 10^5 in the spherical $(0, 1, 1, 2)$ packing

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